Algorithms

Lecture11

Hash Tables

- Direct-address tables
- Hash tables
- Hash functions
- Open addressing

Introduction

- Many applications require a dynamic set that supports only the **dictionary operations** INSERT, SEARCH, and DELETE.
- A hash table is effective for implementing a dictionary.
- The expected time to search for an element in a hash table is *O*(1), under some reasonable assumptions.
- Worst-case search time is O(*n*), however.
- A hash table is a generalization of an ordinary array.
- With an ordinary array, we store the element whose key is *k* in position *k* of the array.
- Given a key *k*, we find the element whose key is *k* by just looking in the *k*th position of the array. This is called **direct addressing**.

Introduction

- We use a hash table when we do not want to (or can't) allocate an array with one position per possible key.
- Use a hash table when the number of keys actually stored is small relative to the number of possible keys.
- A hash table is an array, but it typically uses a size proportional to the number of keys to be stored (rather than the number of possible keys).
- Given a key *k*, don't just use *k* as the index into the array. Instead, compute a function of *k*, and use that value to index into the array. We call this function a **hash function**.

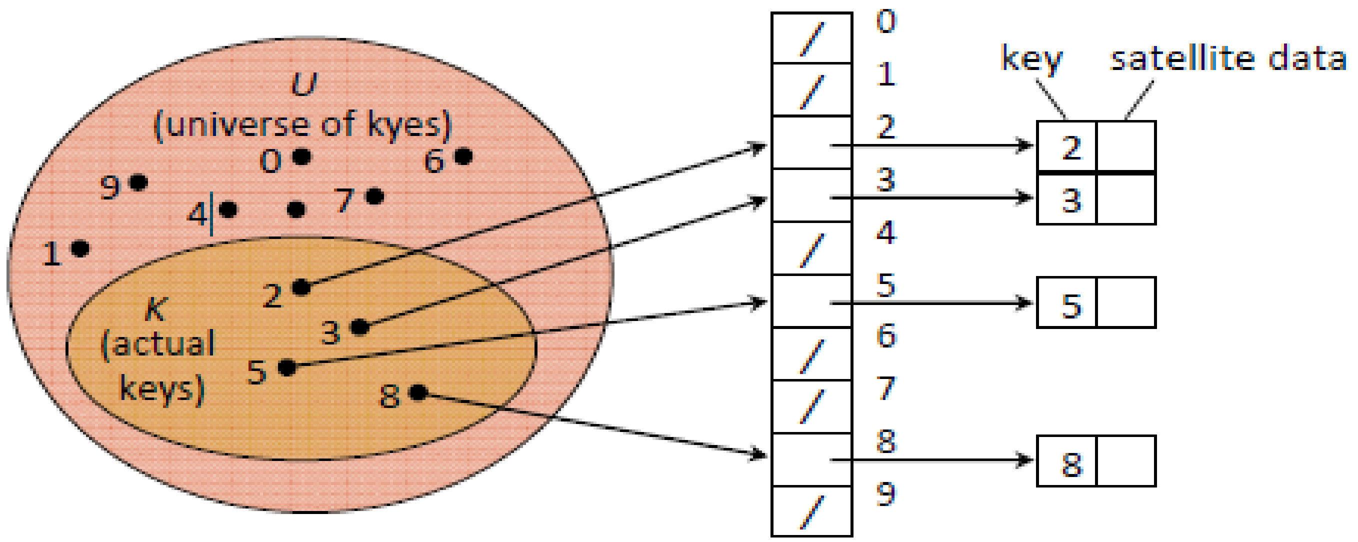
Introduction

- Issues that we'll explore in hash tables:
- How to compute hash functions?
  - The multiplication methods.
  - The division methods.

- What to do when the hash function maps multiple keys to the same table entry? (collision)
  - Chaining.
  - Open addressing.

Direct-address tables

- Maintain a dynamic set.
- Each element has a key drawn from a universe $U = \{0, 1, \dots, m - 1\}$ where m isn't too large.
- No two elements have the same key.
- Represent by **direct-address table**, or array, $T[0..m\text{-}1]$:
  - Each **slot**, or position, corresponds to a key in U.
  - If there is an element x with key k, then $T[k]$ contains a pointer to x.
  - Otherwise, $T[k]$ is empty, represented by NIL.



- Dictionary operations are trivial and take *O*(1) time each:

```
DIRECT-ADDRESS-SEARCH(T, k)
    return T[k]
DIRECT-ADDRESS-INSERT(T, x)
    T[key[x]] ← x
DIRECT-ADDRESS-DELETE(T, x)
    T[key[x]] ← NIL
```

Problem:

- If the universe *U* is large, storing a table of size $|U|$ may be impractical or impossible.
- The set *K* of keys actually stored is small, compared to *U*, so that most of the space allocated for *T* is wasted.

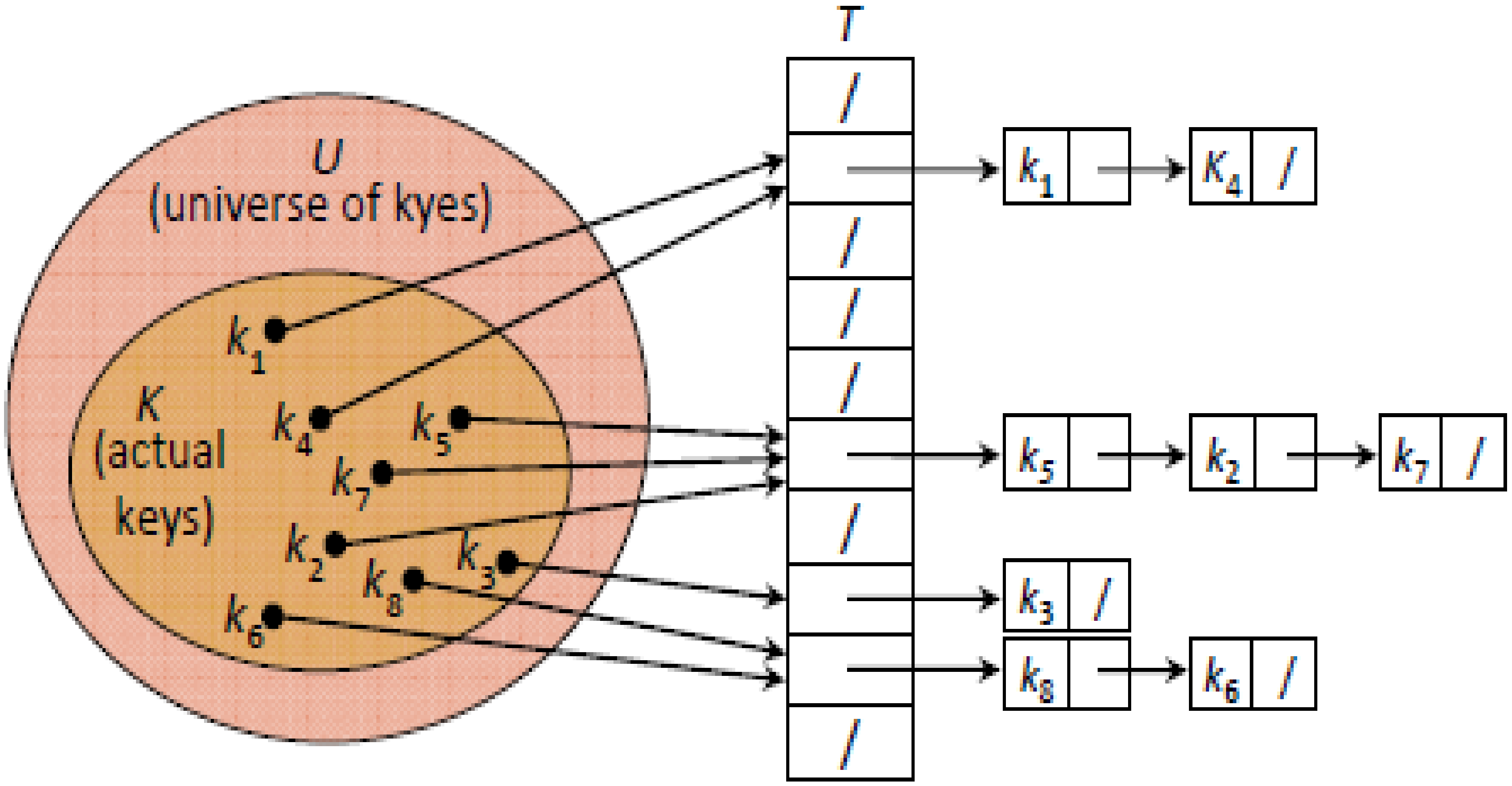
Solution: Hash tables

- When *K* is much smaller than *U*, a hash table requires much less space than a direct-address table.
- Storage requirements can be reduced to $\Theta(|K|)$.
- Searching for an element requires $O(1)$ time, but in the **average case**, not the **worst case**.

Hash Tables

- **Idea:** Instead of storing an element with key *k* in slot *k*, use a function *h* and store the element in slot *h*(*k*).
- We call *h* a **hash function**.
- $h : U \ \ \{0, 1, . . . , m - 1\}$, so that *h*(*k*) is a legal slot number in *T*.
- We say that *k* **hashes** to slot *h*(*k*).
- We also say that *h*(*k*) is the **hash value** of key *k*.

Hash Tables



Collisions: When two or more keys hash to the same slot.

- Can happen when there are more possible keys than slots ($|U| > m$).

Methods to resolve the collision problem.

- **Chaining**
- **Open addressing**

- Chaining is usually better than open addressing.

Collision resolution by chaining

- Put all elements that hash to the same slot into a linked list.
- Slot *j* contains a pointer to the head of the list of all stored elements that hash to *j*.
- If there are no such elements, slot *j* contains NIL.

Dictionary Operations

How to implement dictionary operations with chaining:

*CHAINED-HASH-**INSERT**(T,x):*

Insert x at the head of list T[h(key[x])]

- Worst-case running time is *O*(1).
- Assumes that the element being inserted isn't already in the list.
- It would take an additional search to check if it was already inserted.

*CHAINED-HASH-**SEARCH**(T,k):*

Search for an element with key k in list T[h(k)]

- Running time is proportional to the length of the list of elements in slot *h*(*k*).

Dictionary Operations….

*CHAINED-HASH-**DELETE**(T,x):*

Delete x from the list T[h(key[x])]

- Given pointer *x* to the element to delete, so no search is needed to find this element.
- Worst-case running time is *O*(1) time if the lists are doubly linked.
- If the lists are singly linked, then deletion takes as long as searching, because we must find *x*’s predecessor in its list.

Analysis of hashing with chaining

Given a key, how long does it take to find an element with that key?

Analysis is in terms of the **load factor** $\alpha = n / m$:

- n = # of elements in the table.
- m = # of slots in the table = # of (possibly empty) linked lists.
- Load factor is average number of elements per linked list.
- Can have $\alpha < 1$, $\alpha = 1$, or $\alpha > 1$.

- **Worst case** is when all n keys hash to the same slot
- get a single list of length n
- worst-case time to search is $\Theta(n)$, plus time to compute hash function.
- **Average case** depends on how well the hash function distributes the keys among the slots.`

Average-case performance

- Assume **simple uniform hashing**: any given element is equally
- likely to hash into any of the m slots.
- For $j = 0, 1, \ldots, m-1$, denote the length of the list $T[j]$ by nj, so
- that $n = n0 + n1 + \ldots + nm-1$.
- Average value of nj is E[nj] = ⍰ = n/m.
- Assume that the hash value $h(k)$ can be computed in $O(1)$ time.
- Time for the element with key k depends on the length $nh(k)$
- of the list $T[h(k)]$.
- We consider two cases:
  - contains no element with key k $\longrightarrow$ unsuccessful.
  - contain an element with key k $\longrightarrow$ successful.

Average-case performance

- Assume **simple uniform hashing**: any given element is equally likely to hash into any of the m slots.
- For $j = 0, 1, \ldots, m-1$, denote the length of the list $T[j]$ by n_j, so
- that $n = n_0 + n_1 + \ldots + n_{m-1}$.
- Average value of n_j is $E[n_j] = \alpha = n/m$.
- Assume that the hash value $h(k)$ can be computed in $O(1)$ time.

- Time for the element with key k depends on the length $n_{h(k)}$ of the list $T[h(k)]$.

- We consider two cases:
  - contains no element with key k $\rightarrow$ unsuccessful.
  - contain an element with key k $\rightarrow$ successful.

Theorem 11.1

- An **unsuccessful search** takes expected timeᶱ (1+ α).

Proof:

- Under the assumption of simple uniform hashing, any key not already in the table is equally likely to hash to any of the *m* slots.
- To search unsuccessfully for any key *k*, need to search to the end of the list *T*[*h*(*k*)].
- This list has expected length E[*nh*(*k*)] = α.
- Therefore, the expected number of elements examined in an unsuccessful search is .
- Adding in the time to compute the hash function.
- The total time required isᶱ (1 +α).

Theorem 11.2

- An **successful search** takes expected time Θ (1+ α).

Proof:

- Assume the element being searched for is equally likely to be any of the *n* elements in the table *T*.
- During a successful search for *x*, the # of elements examined = # of elements in the list before *x* + 1.
- The expected length of that list is (*n* – *i*)/*m*.
- The expected # of elements examined in a successful search is

$$\frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{n-i}{m}\right)=1+\frac{1}{nm}\sum_{i=1}^{n}(n-i)=1+\frac{1}{nm}\left(\frac{n(n-1)}{2}\right)=1+\frac{\alpha}{2}-\frac{\alpha}{2n}.$$

- The total time is Θ(2 + α /2 – α /2*n*) = Θ (1+ α).